

# Quantum Inequalities in Curved Two Dimensional Spacetimes

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## Abstract

In quantum field theory there exist states for which the energy density is negative. It is important that these negative energy densities satisfy constraints, such as quantum inequalities, to minimize possible violations of causality, the second law of thermodynamics, and cosmic censorship. In this paper I show that conformally invariant scalar and Dirac fields satisfy quantum inequalities in two dimensional spacetimes with a conformal factor that depends on  $x$  only or on  $t$  only. These inequalities are then applied to two dimensional black hole and cosmological spacetimes. It is shown that the bound on the negative energies diverges to minus infinity as the event horizon or initial singularity is approached. Thus, neglecting back reaction, negative energies become unconstrained near the horizon or initial singularity. The results of this paper also support the hypothesis that the quantum interest conjecture applies only to deviations from the vacuum polarization energy, not to the total energy.

# Introduction

In classical physics it is expected that the energy-momentum tensor will satisfy the weak energy condition  $T_{\mu\nu}V^\mu V^\nu \geq 0$  for all timelike vectors  $V^\mu$  (see [1] for exceptions). This condition ensures that all observers measure a positive energy density. However, in quantum field theory there exist states for which the expectation value of the energy-momentum tensor violates the weak energy condition. Such exotic matter is of interest since it is required to maintain wormholes [2, 3, 4, 5, 6, 7] and to create warp drives [8, 9].

Over the past decade Ford and Roman [10, 11, 12] have studied the properties of exotic matter extensively. In four dimensional flat spacetime they have shown that massless bosonic fields satisfy the quantum inequalities

$$\hat{\rho} = \frac{t_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{tt} \rangle}{(t^2 + t_0^2)} dt \geq -\frac{3A}{32\pi^2 t_0^4} \quad (1)$$

where  $\langle T_{tt} \rangle$  is the expectation value of the normal ordered energy density,  $A = 1$  for massless scalar fields, and  $A = 2$  for the electromagnetic field. The quantity  $\hat{\rho}$  samples  $\langle T_{tt} \rangle$  over a time interval of order  $t_0$  using the sampling function  $h(t) = t_0/[\pi(t^2 + t_0^2)]$ . These quantum inequalities show that an observer can measure a negative energy density for a time  $\lesssim |\rho|^{-1/4}$ . In two dimensional flat spacetime they found that

$$\hat{\rho} \geq -\frac{1}{8\pi t_0^2} \quad (2)$$

for a massless scalar field. The bounds found by Ford and Roman are not optimal bounds. In two dimensional flat spacetime Flanagan [13] has shown that the optimal bound for a scalar field is given by

$$\hat{\rho} \geq -\frac{1}{24\pi} \int_{-\infty}^{\infty} \frac{h'(t)^2}{h(t)} dt \quad (3)$$

where  $h(t)$  is any sampling function that satisfies  $h(t) \geq 0$  and  $\int_{-\infty}^{\infty} h(t) dt = 1$ . Non-optimal quantum inequalities have also been found for scalar fields in static Robertson-Walker, de Sitter, and Schwarzschild spacetimes [17, 18, 19].

Some work has also been done on negative energy density states for the Dirac equation in four dimensional Minkowski space. In an earlier paper [14] I examined a class of states that produced violations of the weak energy condition and showed that they satisfied a quantum inequality of the form (1).

In this paper I show that (2) is also an optimal bound for the massless Dirac field in two dimensional flat spacetimes. I also find the optimal bound for massless Dirac and scalar fields in two dimensional curved spacetimes with a conformal factor dependent on  $x$  only or on  $t$  only. These results are then applied to two dimensional black hole and cosmological spacetimes. It is shown that the bound on the negative energies diverges to

minus infinity (neglecting back reaction) as the event horizon or cosmological singularity is approached. Thus, the negative energies become unconstrained near the horizon or the initial singularity.

The quantum inequalities derived in this paper can also be used to support the quantum interest conjecture [15]. According to this conjecture any negative energy flux must be preceded or followed by a larger positive energy flux (i.e. the negative flux must be repaid with interest by the positive flux). Pretorius [16] has shown that the quantum interest conjecture for scalar fields in Minkowski space follows from the scaling properties of the corresponding quantum inequalities. Since the Dirac equation satisfies the same quantum inequalities in two dimensional Minkowski space it will also satisfy the quantum interest conjecture. Pretorius also conjectured that in curved spacetime the quantum interest conjecture applies only to deviations from the ground state, not to the total energy. This conjecture is shown to be true for the spacetimes examined in this paper. I will take  $\hbar = c = G = 1$  throughout this paper.

## The Energy-Momentum Tensor

In any two dimensional spacetime the metric can be taken to have the conformally flat form

$$ds^2 = C(x, t)[dt^2 - dx^2]. \quad (4)$$

In null coordinates ( $u = t - x$ ,  $v = t + x$ ) the metric becomes

$$ds^2 = C(u, v)dudv. \quad (5)$$

The conservation laws  $\nabla_\mu T^\mu_\nu = 0$  give

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} [\sqrt{g} T^\alpha_\beta] = \frac{1}{2} T \partial_\beta (\ln C) \quad (6)$$

where  $T = T^\alpha_\alpha$  and  $T^\alpha_\beta$  is the energy-momentum tensor. Using  $T^u_u = T^v_v = \frac{1}{2}T$  and  $T_{uu} = \frac{1}{2}CT^v_u$ ,  $T_{vv} = \frac{1}{2}CT^u_v$  we find that

$$\partial_v T_{uu} = -\frac{1}{4}C\partial_u T \quad (7)$$

and

$$\partial_u T_{vv} = -\frac{1}{4}C\partial_v T. \quad (8)$$

Thus in flat spacetime, where the conformal anomaly vanishes (i.e.  $T = 0$ ), a conformally invariant theory, such as the massless scalar or Dirac field, will have  $T_{uu} = T_{uu}(u)$ ,  $T_{vv} = T_{vv}(v)$ , and  $T_{uv} = 0$ . This can be seen explicitly for the massless scalar and Dirac fields.

The scalar field equation  $\square^2\phi = 0$  has the general solution  $\phi(u, v) = \phi_u(u) + \phi_v(v)$  and the energy-momentum tensor has the components  $T_{uu} = (\partial_u\phi_u)^2$ ,  $T_{vv} = (\partial_v\phi_v)^2$ , and  $T_{uv} = 0$ .

The massless Dirac equation

$$i\gamma^\mu\partial_\mu\psi = 0 \quad (9)$$

with

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (10)$$

and

$$\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (11)$$

has the general solution

$$\psi = \begin{bmatrix} \psi_u(u) \\ \psi_v(v) \end{bmatrix}. \quad (12)$$

The energy-momentum tensor is given by

$$T_{uu} = \frac{i}{2}[\psi_u^\dagger\partial_u\psi_u - (\partial_u\psi_u^\dagger)\psi_u], \quad (13)$$

$$T_{vv} = \frac{i}{2}[\psi_v^\dagger\partial_v\psi_v - (\partial_v\psi_v^\dagger)\psi_v], \quad (14)$$

and  $T_{uv} = 0$ . Note that  $\psi$  can be taken to be real since the gamma matrices are purely imaginary. For a discussion of the 1+1 dimensional Dirac equation see chapter four of Green, Schwarz, and Witten [20]. Both of these energy-momentum tensors satisfy  $T_{uu} = T_{uu}(u)$ ,  $T_{vv} = T_{vv}(v)$ , and  $T_{uv} = 0$ .

## Quantum Inequalities

Consider two conformally related spacetimes, one with metric

$$ds^2 = C(x, t)[dt^2 - dx^2] \quad (15)$$

and the other one with  $C(x, t) = 1$ . The energy-momentum tensor for scalar and Dirac fields satisfies [21, 22, 23, 24, 25]

$$\langle T_{\alpha\beta}^{(C)} \rangle = \langle T_{\alpha\beta}^{(\eta)} \rangle + \Theta_{\alpha\beta} - \frac{1}{48\pi}RC\eta_{\alpha\beta} \quad (16)$$

where  $T_{\alpha\beta}^{(C)}$  is the renormalized energy-momentum tensor with the metric given in (15),  $T_{\alpha\beta}^{(\eta)}$  is the energy-momentum tensor with  $C(x, t) = 1$ ,

$$\Theta_{uu} = -\frac{1}{12\pi}C^{\frac{1}{2}}\partial_u^2C^{-\frac{1}{2}}, \quad (17)$$

$$\Theta_{vv} = -\frac{1}{12\pi} C^{\frac{1}{2}} \partial_v^2 C^{-\frac{1}{2}}, \quad (18)$$

$$\Theta_{uv} = \Theta_{vu} = 0, \quad (19)$$

and  $R = \square^2 \ln C$  is the Ricci scalar.

First consider the quantum inequalities in a two dimensional flat spacetime. The metric is

$$ds^2 = dt^2 - dx^2 = dudv. \quad (20)$$

Now consider the conformally related spacetime

$$ds^2 = f'(v) dudv = dudV \quad (21)$$

where  $V = f(v)$ . From (16) we find that

$$\langle T_{vv}^{(f')} \rangle = \langle T_{vv}^{(\eta)} \rangle + \Theta_{vv}, \quad (22)$$

which gives

$$(f')^2 \langle T_{VV} \rangle = \langle T_{vv}^{(\eta)} \rangle + \Theta_{vv}, \quad (23)$$

where  $T_{VV}$  is  $VV$  component of the energy-momentum tensor in the  $(u, V)$  coordinate system. Now multiply by the sampling function  $h(v)$  and integrate over  $v$  to get

$$\int_{-\infty}^{\infty} f'(v) h(v) \langle T_{VV} \rangle f'(v) dv = \int_{-\infty}^{\infty} h(v) \langle T_{vv}^{(\eta)} \rangle dv + \int_{-\infty}^{\infty} h(v) \Theta_{vv} dv. \quad (24)$$

Choose  $f'(v)$  such that  $f'(v)h(v) = 1$ . This gives

$$\int_{-\infty}^{\infty} \langle T_{VV} \rangle dV = \int_{-\infty}^{\infty} h(v) \langle T_{vv}^{(\eta)} \rangle dv + \Delta \quad (25)$$

where

$$\Delta = \frac{1}{48\pi} \int_{-\infty}^{\infty} \frac{h'(v)^2}{h(v)} dv. \quad (26)$$

Note that I have taken  $h'(v) \rightarrow 0$  as  $v \rightarrow \pm\infty$  to obtain (26). Now the left hand side of (25) is greater than or equal to zero since it is the normal ordered Hamiltonian for the left moving sector. Thus

$$\int_{-\infty}^{\infty} h(v) \langle T_{vv}(v) \rangle dv \geq -\Delta. \quad (27)$$

This is an optimal bound since the equality can be reached by using the ground state of the left moving sector. A similar result holds for  $\int_{-\infty}^{\infty} h(u) \langle T_{uu}(u) \rangle du$ . Thus from  $T_{tt} = T_{uu} + T_{vv}$  we find the optimal inequality (at fixed  $x$ )

$$\hat{\rho} = \int_{-\infty}^{\infty} h(t) \langle T_{tt}(t) \rangle dt \geq -\frac{1}{24\pi} \int_{-\infty}^{\infty} \frac{h'(t)^2}{h(t)} dt \quad (28)$$

for scalar and Dirac fields in two dimensional flat spacetime. This is the result obtained by Flanagan [13] for scalar fields. As discussed in the introduction, the above quantum inequality implies that the Dirac field satisfies the quantum interest conjecture in two dimensional Minkowski space.

Now consider a two dimensional curved spacetime with  $C = C(x)$ . An observer with velocity  $V^\alpha$  will measure

$$\rho = \langle T_{\alpha\beta} \rangle V^\alpha V^\beta. \quad (29)$$

for the expectation value of the energy density. From (16) we find

$$\rho_{(C)} = \frac{1}{C} \rho_{(\eta)} + \frac{1}{C} \left[ \Theta_{\alpha\beta} V_{(\eta)}^\alpha V_{(\eta)}^\beta - \frac{RC}{48\pi} \right] \quad (30)$$

where  $\rho_{(C)}$  is the energy density in the spacetime,  $\rho_{(\eta)}$  is the energy density in the conformally related flat spacetime, and  $V_{(\eta)}^\alpha$  is the four velocity of the observer in the flat spacetime (note that  $V_{(C)}^\alpha = C^{-\frac{1}{2}} V_{(\eta)}^\alpha$ ).

Now consider an observer at rest in the  $(x, y)$  coordinate system. Multiply (30) by  $h_C(t)\sqrt{C}$  and integrate to get

$$\int_{-\infty}^{\infty} \rho_{(C)} h_C \sqrt{C} dt = \frac{1}{C} \int_{-\infty}^{\infty} \rho_{(\eta)} h_\eta dt - \hat{\Delta} \quad (31)$$

where  $h_\eta = \sqrt{C} h_C$  and

$$\hat{\Delta} = -\frac{1}{C} \left[ \Theta_{tt} - \frac{RC}{48\pi} \right]. \quad (32)$$

Note that  $\int_{-\infty}^{\infty} h_C \sqrt{C} dt = \int_{-\infty}^{\infty} h_\eta dt = 1$ . Using  $R = \square^2 \ln C$  and  $\Theta_{tt} = -\frac{1}{24\pi} C^{\frac{1}{2}} \partial_x^2 (C^{-\frac{1}{2}})$  gives

$$\hat{\Delta} = \frac{1}{6\pi C} [C^{\frac{1}{4}} \partial_x^2 (C^{-\frac{1}{4}})]. \quad (33)$$

Thus from (28), (31), and (33) we find that

$$\hat{\rho}_C \geq -\frac{1}{24\pi C} \left[ \int_{-\infty}^{\infty} \frac{h'_\eta(t)^2}{h_\eta(t)} dt + 4C^{\frac{1}{4}} \partial_x^2 (C^{-\frac{1}{4}}) \right]. \quad (34)$$

This is an optimal bound since the equality can be reached by using the state that minimizes  $\int_{-\infty}^{\infty} \rho_{(\eta)} h_\eta dt$ . The first term on the right hand side of (34) will satisfy the scaling arguments used by Pretorius [16], but the second term (i.e. the vacuum polarization term) does not. Thus, the quantum interest conjecture applies to deviations from the vacuum polarization energy, not to the total energy.

A similar bound can be derived for spacetimes with  $C = C(t)$ . In this case we will average over space instead of time. It is easy to show that (28) becomes

$$\tilde{\rho} = \int_{-\infty}^{\infty} h(x) \langle T_{tt} \rangle dx \geq -\frac{1}{24\pi} \int_{-\infty}^{\infty} \frac{h'(x)^2}{h(x)} dx \quad (35)$$

and that (34) becomes

$$\tilde{\rho}_C \geq -\frac{1}{24\pi C} \left[ \int_{-\infty}^{\infty} \frac{h'_\eta(x)^2}{h_\eta(x)} dx + \frac{1}{2} \left( \frac{\partial_t C}{C} \right)^2 \right]. \quad (36)$$

This is also an optimal bound since the equality can be reached by using the state that minimizes  $\int_{-\infty}^{\infty} \rho_\eta h_\eta dx$ . From the above inequalities we see that the right hand side will generally diverge as  $C \rightarrow 0$ . Thus, neglecting back reaction, the negative energy densities become unconstrained in regions where  $C \rightarrow 0$ .

## Black Hole and Cosmological Spacetimes

Consider the two dimensional black hole spacetime

$$ds^2 = \left(1 - \frac{2m}{\bar{r}}\right) dt^2 - \left(1 - \frac{2m}{\bar{r}}\right)^{-1} d\bar{r}^2. \quad (37)$$

In conformally flat coordinates

$$ds^2 = \left[1 - \frac{2m}{\bar{r}(r)}\right] (dt^2 - dr^2) \quad (38)$$

where  $\bar{r}(r)$  is defined via

$$r = \bar{r} + 2m \ln \left| \frac{\bar{r}}{2m} - 1 \right|. \quad (39)$$

Note that the horizon is at  $\bar{r} = 2m$  and at  $r \rightarrow -\infty$ .

Now consider an observer at rest near the horizon. Close to the horizon

$$C(r) \simeq e^{r/2m} \quad (40)$$

and

$$\hat{\rho}_C \geq -\frac{1}{24\pi C} \left[ \int_{-\infty}^{\infty} \frac{h'_\eta(t)^2}{h_\eta(t)} dt + \frac{1}{16m^2} \right]. \quad (41)$$

Thus, neglecting back reaction, as the horizon is approached the right hand side diverges and the quantum inequality does not constrain the negative energy densities.

Consider a specific example using the sampling function

$$h_C(t) = \frac{\tau_0}{\pi[Ct^2 + \tau_0^2]}, \quad (42)$$

which has been used by Ford and Roman in the flat spacetime case with  $C = 1$ . Note that  $\sqrt{C}t$  is the proper time measured by the observer. The function has a maximum height of  $1/(\pi\tau_0)$  and a proper width of  $\sim \tau_0$ . The flat spacetime sampling function is

$$h_\eta = \frac{\sqrt{C}\tau_0}{\pi[Ct^2 + \tau_0^2]}. \quad (43)$$

It has a maximum height of  $\sqrt{C}/(\pi\tau_0)(\rightarrow 0 \text{ as } \bar{r} \rightarrow 2m)$  and a proper width of  $\sim \tau_0/\sqrt{C}(\rightarrow \infty \text{ as } \bar{r} \rightarrow 2m)$ . The inequality (41) becomes

$$\hat{\rho}_C \geq -\frac{1}{48\pi C} \left[ \frac{C}{\tau_0^2} + \frac{1}{8m^2} \right]. \quad (44)$$

Let's see how the Boulware state near the horizon compares with the above inequality. Since the Boulware state diverges at the event horizon it will not be the quantum state outside a black hole. It is instead the quantum state exterior to a static mass distribution that is larger than its Schwarzschild radius. I will take the mass distribution to be only slightly larger than its Schwarzschild radius (of course, the matter will be under arbitrarily large stresses which is physically unrealistic). For a metric of the form

$$ds^2 = f(\bar{r})dt^2 - \frac{1}{f(\bar{r})}d\bar{r}^2 \quad (45)$$

the energy density in the Boulware state is given by [26]

$$\rho_B = \frac{1}{24\pi} \left[ f'' - \frac{(f')^2}{4f} \right]. \quad (46)$$

Note that this will be the energy density in the  $(r, t)$  coordinate system since  $\rho = -T^t_t$  is unaffected by changes in the radial coordinate. For  $f = (1 - 2m/\bar{r})$  we have

$$\rho_B = -\frac{1}{384\pi m^2} \left( 1 - \frac{2m}{\bar{r}} \right)^{-1} \quad (47)$$

near the horizon. This is just the second term on the left hand side of (41). Thus the Boulware state satisfies the quantum inequality (41), as it must.

Now consider the cosmological spacetime

$$ds^2 = t^n [dt^2 - dx^2] \quad n > 0. \quad (48)$$

The last term on the right hand side of (36) is

$$-\frac{1}{48\pi C} \left( \frac{\partial_t C}{C} \right)^2 = -\frac{n^2}{48\pi} t^{-(n+2)} \quad (49)$$

which diverges as  $t \rightarrow 0$ . Thus as we approach the initial singularity the negative energy densities become unconstrained. Of course, as one approaches the initial singularity quantum gravity effects are expected to become important and the above analysis will break down.



## Conclusion

Optimal quantum inequalities were derived for scalar and Dirac fields in two dimensional spacetimes in which the conformal factor is a function of  $x$  only or of  $t$  only. For spacetimes with a metric of the form

$$ds^2 = C(x)(dt^2 - dx^2) \quad (50)$$

it was shown that the energy density satisfies the optimal bound

$$\hat{\rho} = \int_{-\infty}^{\infty} h(t) \langle T_{tt} \rangle dt \geq -\frac{1}{24\pi C} \left[ \int_{-\infty}^{\infty} \frac{h'(t)^2}{h(t)} dt + 4C^{\frac{1}{4}} \partial_2^x (C^{-\frac{1}{4}}) \right] \quad (51)$$

where  $h(t)$  is a sampling function that satisfies  $h(t) \geq 0$  and  $\int_{-\infty}^{\infty} h(t) dt = 1$ . A similar quantum inequality was also found when the conformal factor is a function of time only. These inequalities were then applied to black hole and cosmological spacetimes. It was shown that the bound on the negative energies diverges to minus infinity as the horizon or initial singularity is approached. Thus, neglecting back reaction, the negative energy densities become unconstrained near the horizon or initial singularity.

I also showed that the quantum interest conjecture holds for scalar and Dirac fields in two dimensional Minkowski space and that in curved spacetimes this conjecture applies to deviations from the vacuum polarization energy, not to the total energy.

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